N-widths on the classes of multivariate bandlimited functions*

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Abstract In this paper, the tight order on the n-widths of the classes of multivariate bandlimited functions in $L_p(\mathbb{R}^d)$, $1<p<\infty$ are determined.

Keywords: n-width, multivariate bandlimited function, sampling theorem.

First, we recall some definitions. Let X be a normed linear space, and X_n an n-dimensional subspace of X and A a subset of X, then the deviation $E(A; X_n)$ of A from X_n is defined by

$$E(A;X_n)=\sup_{x\in A}\inf_{y\in X_n}\|x-y\|.$$

This deviation provides information on how well A may be approximated by elements of X_n. However, another choice of X_n might provide a smaller deviation. Thus, we consider the possibility of allowing the n-dimensional subspaces to vary within X. This idea, introduced by Kolmogorov[1], is now referred to as the n-width, in the sense of Kolmogorov or as the n-width of A in X, which is given by

$$d_n(A;X)=\inf_{X_n\subset X}E(A;X_n).$$

If the infimum is attained, then the corresponding X_n is called an optimal subspace.

The calculation of n-widths for various classical spaces of functions plays an important role in the numerical analysis, computational complexity of information bases[2] and approximation theory, since this problem has close relations to many optimal problems, such as ε-complexity of integration and approximation, optimal differentiation, and optimal approximate solutions of the classes of operator equations. The theory of width was proposed by the paper of Kolmogorov[1], and the exact estimate of the width on the one dimensional classical Sobolev classes of smooth functions $W_p^r(([0,1])$ in the space $L_q([0,1])$ was determined for $p=q=2$. Later the asymptotic behavior of the same width for $1\leqslant p, q\leqslant\infty$ was described by Tikhomirov[3], Ismagilov[4], Maiorov[5], Kashin[6] and so on.

Moreover, in the case of multivariate, Höllig[7] investigated the n-width of multivariate Sobolev classes $W_p^r([0,1]^d)$ in $L_q([0,1]^d)$, where the Sobolev space $W_p^r([0,1]^d)$ consists of all functions f such that its generalized derivative $D^\lambda f=D^{|\lambda|}f/D^{\lambda_1}x_1\cdots D^{\lambda_d}x_d$ belongs to the Lebesgue space $L_p([0,1]^d)$ for each $\lambda=(\lambda_1,\cdots,\lambda_d)$, $\lambda_j\in\mathbb{Z}$, $\lambda_j\geqslant 0$, $|\lambda|:=\sum_{i=1}^{d}\lambda_i$, with norm

$$\|f\|_{W_p^r[0,1]^d}:=\|f\|_p+\|f\|_{L_p^r[0,1]^d},$$

where

$$\|f\|_{L_p^r[0,1]^d}:=\sum_{|\lambda|=r}\|f^{(\lambda)}\|_{L_p[0,1]}.$$

Recently, Kudryavtsev[8] determined the tight orders of n-width of $W_p^rH^\omega([0,1]^d)$ in $L_q([0,1]^d)$, where ω is a given modulus of continuity.

For more details about the history of n-width of various classes of functions in different case setting of computation, see References [2, 9~13].

1 Preliminary and main results

The following simple properties of Kolmogorov width d_n is needed in this paper.

Theorem A.[9] Let X_{n+1} be any $(n+1)$-dimensional subspace of a normed linear space X and denote by $\lambda S(X_{n+1})$ a ball of radius λ in X_{n+1}, then

$$d_n(\lambda S(X_{n+1});X)=\lambda.$$

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Lemma 1. [9] a) Let X be a normed linear space and $A \subseteq X$, then

$$d_{n+1}(A; X) \leqslant d_n(A; X).$$

b) Let X be a normed linear space and $B \subseteq A \subseteq X$, then

$$d_n(B; X) \leqslant d_n(A; X).$$

Let $v = \{v_1, v_2, \cdots, v_d\} \in \mathbb{R}_+^d$, and let $E_v(\mathbb{R}^d)$ be the class of all entire functions of exponential type v. Denote by $B_v(\mathbb{R}^d)$ the subspace of $E_v(\mathbb{R}^d)$ bounded on $\mathbb{R}^d$, and write

$$B_{v,p}(\mathbb{R}^d) := B_v(\mathbb{R}^d) \cap L_p(\mathbb{R}^d),$$
$$1 \leqslant p \leqslant \infty, \quad B_{v,\infty}(\mathbb{R}^n) := B_v(\mathbb{R}^n),$$

when $p = 2$, and $B_{v,2}(\mathbb{R}^d) = PW_v^d$ is the classical Paley-Wiener space. Since these spaces are very important in both mathematical theory and engineering applications, $B_{v,p}(\mathbb{R}^d)$ has been studied widely (see Refs. [14~18]).

Supposing that $\Delta_v = \{|x_j| \leqslant v_j, x_j \in \mathbb{R}, j = 1, \cdots, d\}$, it follows from the Schwarzt Theorem[19] that

$$B_{v,p}(\mathbb{R}^d) = \{f \in L_p(\mathbb{R}^d): \operatorname{supp} \hat{f} \subset \Delta_v\}.$$

Hence, $B_{v,p}(\mathbb{R}^d)$ is also called the class of bandlimited functions.

The well-known Whittaker-Shannon-Kotelni'kov sampling theorem[16] says that if a function f satisfies $f \in PW_v := PW_w^1$, which is the classical one-dimensional Paley-Wiener space, then it can be reconstructed from its equidistance (regular) samples at the points $x_k = k\pi/v$, $k \in \mathbb{Z}$ on $\mathbb{R}$, via the formula

$$f(x) = \sum_{k=-\infty}^{\infty} f(k\pi/v) \operatorname{sinc} \pi(x - k\pi/v), \quad x \in \mathbb{R},$$

where $\operatorname{sinc} t = t^{-1} \sin t$ if $t \neq 0$ and 1 if $t = 0$, and the series converges absolutely and uniformly on compact sets of the real line $\mathbb{R}$.

Since the sampling theory is one of the most important mathematical tools used in communication engineering and data processing, it has been generalized in many different directions[15, 18, 20]. Especially, this theorem has been generalized to multivariate case by Wang and Fang[17] using real and harmonic analysis. They studied the Whittaker-Shannon-Kotel'nikov multivariate sampling theorem in space $B_{v,p}(\mathbb{R}^d)$, $1 < p \leqslant \infty$, and also proved an inequality of Marcinkiewicz type in the space $B_{v,p}(\mathbb{R}^d)$. These results will be used this paper.

Theorem B. [17] Let $f \in B_{v,p}(\mathbb{R}^d)$, $1 \leqslant p < \infty$, then

$$\text{(a) } f(\boldsymbol{x}) = \sum_{k \in \mathbb{Z}^d} f(\boldsymbol{k}\pi/v) \prod_{j=1}^{d} \operatorname{sinc} v_j(x_j - k_j\pi/v_j).$$

In the above equation, the series on the right side uniformly converges on $\mathbb{R}^d$.

(b) Put $h_j = \pi/v_j$, $j = 1, \cdots, d$, there is a constant $C_{p,d}$ only depending on p and d, such that

$$\| f \|_{p(\mathbb{R}^d)} \leqslant C_{p,d} \left(h_1 \cdots h_d \sum_{k \in \mathbb{Z}^d} |f(\boldsymbol{k}\pi/v)|^p \right)^{\frac{1}{p}},$$

where $\boldsymbol{k} = (k_1, \cdots, k_d) \in \mathbb{Z}^d$, $\boldsymbol{k}\pi/v = (k_1\pi/v_1, \cdots, k_d\pi/v_d)$.

Theorem C. [19] Let $f \in B_{v,p}(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, then

$$C_{p',d} \left(h_1 \cdots h_d \sum_{k \in \mathbb{Z}^d} |f(k\pi/v)|^p \right)^{\frac{1}{p}} \leqslant \| f \|_{p(\mathbb{R}^d)},$$

where the constant $C_{p',d}$ only depends on p.

We need also two notations of the vectors in $\mathbb{R}^d$. Suppose $\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, and let

$$\boldsymbol{x}^1 := x_1 + \cdots + x_d,$$
$$|\boldsymbol{x}| := |x_1| + \cdots + |x_d|.$$

Now we give our main results.

Theorem 1. Let $E_{v,m} = \{f: \| \boldsymbol{x}^m f(x) \|_{p(\mathbb{R}^d)} \leqslant 1, f \in B_{v,p}(\mathbb{R}^d), 1 < p < \infty\}$, $m \in \mathbb{N}$, then

$$C_2 \bar{h}^{-m} N^{-\frac{m}{d}} \leqslant d_N(E_{v,m}; L_p(\mathbb{R}^d)) \leqslant C_1 h_0^{-m} N^{-\frac{m}{d}},$$

where $\bar{h} = \max\{h_1, \cdots, h_d\}$, $h_0 = \min\{h_1, \cdots, h_d\}$, and the constants C_1 and C_2 only depend on p, and d.

Moreover, in the case of $p = 2$, we have the following exact estimates.

Theorem 2. If $v_1 = \cdots = v_d := \pi/h$, then for all $(2n-1)^d \leqslant N < (2n+1)^d$, we have

$$d_N(E_{v,m}; L_2(\mathbb{R}^d)) = h^{-m} N^{-\frac{m}{d}} = (\pi/v)^{-m} N^{-\frac{m}{d}}.$$

Remark 1. The exact estimate in the case of one dimension in Theorem 2 was determined by Jagerman[20] (see also Ref. [9]):

$$d_{2N-1}(E_{v,m}; L_2(\mathbb{R})) = d_{2N}(E_{v,m}; L_2(\mathbb{R})) = (\pi/v)^{-m}N^{-m}, \quad n = 1, 2, \cdots.$$

2 Proof of main results

The following lemma gives a Parseval type equality and will be used in the proof of Theorem 2.

Lemma 2.[19] Let $f \in B_{v,2}(\mathbb{R}^d)$, then

$$\|f\|_{2(\mathbb{R}^d)} = \left(\pi_1/v_1 \cdots \pi/v_d \sum_{k\in\mathbb{Z}^d} |f(\boldsymbol{k}\pi/v)|^2\right)^{\frac{1}{2}}.$$

Proof. It follows from part (a) of Theorem B, for $f \in B_{v,2}(\mathbb{R}^d)$, we have

$$f(\boldsymbol{x}) = \sum_{k\in\mathbb{Z}^d} f(\boldsymbol{k}\pi/v) \prod_{j=1}^{d} \operatorname{sinc} v_j(x_j - k_j\pi/v_j).$$

Let

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{d} \operatorname{sinc} v_j(x_j - k_j\pi/v_j), \qquad \forall \boldsymbol{k} \in \mathbb{Z}^d$$

then $\phi_{\boldsymbol{k}}(\boldsymbol{j}\pi/v) = \delta_{\boldsymbol{kj}}$ and

$$\int_{\mathbb{R}^d} \phi_{\boldsymbol{k}}(t)\phi_{\boldsymbol{j}}(t)\mathrm{d}t = \pi/v_1 \cdots \pi/v_d\, \delta_{\boldsymbol{kj}}.$$

Hence the expansion of f is also an orthogonal representation for $f \in B_{v,2}(\mathbb{R}^d)$. The proof of Lemma 2 is complete.

Proof of Theorem 1. We begin with the upper estimate. Note that if $f \in E_{v,m}$, then $\boldsymbol{x}^m f(\boldsymbol{x}) \in B_{v,p}(\mathbb{R}^d)$. Hence from Theorem B (a), we have

$$\boldsymbol{x}^m f(\boldsymbol{x}) = \sum_{k\in\mathbb{Z}^d} (\boldsymbol{k}\pi/v)^m f(\boldsymbol{k}\pi/v) \circ \prod_{j=1}^{d} \operatorname{sinc} v_j(x_j - k_j\pi/v_j),$$

$$f(\boldsymbol{x}) = \sum_{k\in\mathbb{Z}^d} f(\boldsymbol{k}\pi/v) \prod_{j=1}^{d} \operatorname{sinc} v_j(x_j - k_j\pi/v_j).$$

Let

$$X_{(2n-1)^d} = \operatorname{span}\{\phi_j(t): |k_j \leqslant n|, j = 1, \cdots, d\},$$

and

$$E_{n-1}f(\boldsymbol{x}) = \sum_{|k_1|\leqslant n-1} \cdots \sum_{|k_d|\leqslant n-1} f(k\pi/v) \circ \prod_{j=1}^{d} \operatorname{sinc} v_j(x_j - k_j\pi/v_j),$$

then $E_{n-1}f \in X_{(2n-1)^d}$, and we have

$$\begin{aligned} E(E_{v,m}; X_{(2n-1)^d}) &= \sup_{f\in E_{v,m}} \inf_{g\in X_{(2n-1)^d}} \|f-g\|_{p(\mathbb{R}^d)} \leqslant \sup_{f\in E_{v,m}} \|f - E_{n-1}f\|_{p(\mathbb{R}^d)} \\ &= \sup \frac{\left\| \left(\sum_{k\in\mathbb{Z}^d} - \sum_{|k_1|\leqslant n-1} \cdots \sum_{|k_d|\leqslant n-1} \right) f(\boldsymbol{k}\pi/v) \prod_{j=1}^{d} \operatorname{sinc} v_j(x_j - k_j\pi/v_j) \right\|_{p(\mathbb{R}^d)}}{\left\| \sum_{k\in\mathbb{Z}^d} \left(\frac{k\pi}{v} \right)^m f(\boldsymbol{k}\pi/v) \prod_{j=1}^{d} \operatorname{sinc} v_j(x_j - k_j\pi/v_j) \right\|_{p(\mathbb{R}^d)}}. \end{aligned} \tag{1}$$

For $f \in B_{v,p}(\mathbb{R}^d)$, we define 2^d terms of $S_r(f)$ as

$$S_r(f) := \sum_{|k_{j_1}|<n} \cdots \sum_{|k_{j_r}|<n} \sum_{|k_{j_r+1}|\geqslant n} \cdots \sum_{|k_{j_d}|\geqslant n} |f(\boldsymbol{k}\pi/v)|^p, \quad r = 1, \cdots, 2^d,$$

especially, when $r = 2^d$, we write it as

$$S_{2^d}(f) := \sum_{|k_1|<n} \cdots \sum_{|k_d|<n} |f(\boldsymbol{k}\pi/v)|^p.$$

Since $\sum_{k\in\mathbb{Z}^d} |f(\boldsymbol{k}\pi/v)|^p$ is the sum of 2^d terms of $S_r(f)$, and $\sum_{k\in\mathbb{Z}^d} |(\boldsymbol{k}\pi/v)|^{pm} |f(\boldsymbol{k}\pi/v)|^p$ is the sum of 2^d terms of $S_j(\boldsymbol{x}^m f)$, we have

$$\sum_{k\in\mathbb{Z}^d} |f(\boldsymbol{k}\pi/v)|^p - S_{2^d}(f) = \sum_{r<2^d} S_j(f),$$

$$\sum_{k\in\mathbb{Z}^d} |\boldsymbol{k}\pi/v|^{pm} |f(\boldsymbol{k}\pi/v)|^p - S_{2^d}(\boldsymbol{x}^m f) = \sum_{r<2^d} S_j(\boldsymbol{x}^m f).$$

Noting that

$$\begin{aligned} S_r(\boldsymbol{x}^m f) &= \sum_{|k_{j_1}|<n} \cdots \sum_{|k_{j_r}|<n} \sum_{|k_{j_r+1}|\geqslant n} \cdots \sum_{|k_{j_d}|\geqslant n} \circ \left|\frac{k\pi}{v}\right|^{pm} |f(\boldsymbol{k}\pi/v)|^p \\ &\geqslant (nh_0)^{pm} S_r(f), \quad r = 1, \cdots, 2^d - 1, \end{aligned} \tag{2}$$

we obtain

$$\sum_{r<2^d} S_r(\boldsymbol{x}^m f) \geqslant (nh_0)^{pm} \sum_{r<2^d} S_r(f). \tag{3}$$

Combining Eq. (3) with Theorem B (b), Theorem C, and Eq. (1) gives

$$E(E_{v,m}; X_{(2n-1)^d}) \leqslant C_1 \sup \left(\frac{\sum_{r<d} S_r(f)}{\sum_{r<d} S_r(\boldsymbol{x}^m f)} \right)^{\frac{1}{p}}$$

$$\leqslant C_1(nh_0)^{-m}.$$

Therefore, we have

$$d_{(2n-1)^d}(E_{v,m};L_p(\mathbb{R}^d))\leqslant C_1(nh_0)^{-m}. \quad (4)$$

Let $N\in\mathbb{N}$ be given, then exists a $n\in\mathbb{N}$ such that $(2n-1)^d\leqslant N<(2n+1)^d$. Then from Eq. (4), we have

$$d_N(E_{v,m};L_p(\mathbb{R}^d))\leqslant C_1h_0^{-m}N^{-\frac{m}{d}}.$$

Thus we obtain the upper estimates of Theorem 1.

Now we turn to proceed the lower estimate. Considering a subspace

$$M_{(2n+1)^d}=\left\{f:\ f(\boldsymbol{x})=\sum_{|k_1|\leqslant n}\cdots\sum_{|k_d|\leqslant n}f(\boldsymbol{k}\pi/v)\right.$$
$$\cdot\prod_{j=1}^{d}\operatorname{sinc}v_j(x_j-k_j\pi/v_j),$$
$$\left.\|f\|_{p(\mathbb{R}^d)}\leqslant C_2(n\hbar)^{-m}\right\}, \quad (5)$$

where $C_2\leqslant/(C_{p',d}d^mC_{p,d})$, $C_{p',d}$ and $C_{p,d}$ are the constants satisfying the inequalities in the part (b) of Theorem B and Theorem C, respectively, then it is clear that $M_{(2n+1)^d}$ is a $(2n+1)^d$ dimensional subspace of $L_p(\mathbb{R}^d)$. Thus from Theorem C, for $f\in M_{(2n+1)^d}$, we have

$$C_{p',d}\left(h_1\cdots h_d\sum_{|k_1|\leqslant n}\cdots\sum_{|k_d|\leqslant n}|f(\boldsymbol{k}\pi/v)|^p\right)^{1/p}$$
$$\leqslant\|f\|_{p(\mathbb{R}^d)}.$$

Combining the above equation with Eq. (5), we have

$$\left(h_1\cdots h_d\sum_{|k_1|\leqslant n}\cdots\sum_{|k_d|\leqslant n}|f(\boldsymbol{k}\pi/v)|^p\right)^{1/p}$$
$$\leqslant\frac{C_2(n\hbar)^{-m}}{C_{p',d}}. \quad (6)$$

From Eq. (6) and Theorem C, we obtain

$$\|\boldsymbol{x}^mf(\boldsymbol{x})\|_{p(\mathbb{R}^d)}\leqslant C_{p,d}\left(h_1\cdots h_d\sum_{|k_1|\leqslant n}\cdots\sum_{|k_d|\leqslant n}\right.$$
$$\left.\cdot|\boldsymbol{k}\pi/v|^{pm}|f(\boldsymbol{k}\pi/v)|^p\right)^{1/p}$$
$$\leqslant d^m(n\hbar)^mC_{p,d}\left(h_1\cdots h_d\sum_{|k_1|\leqslant n}\cdots\right.$$
$$\left.\cdot\sum_{|k_d|\leqslant n}|f(\boldsymbol{k}\pi/v)|^p\right)^{1/p}$$
$$\leqslant d^m(n\hbar)^m\frac{C_{p,d}C_2(n\hbar)^{-m}}{C_{p',d}}$$
$$=1.$$

Therefore $M_{(2n+1)^d}\subseteq E_{v,m}$, and from Eq. (5), Theorem A and Lemma 1, when $(2n-1)^d\leqslant N<(2n+1)^d$, we have

$$d_N(E_{v,m};L_p(\mathbb{R}^d))$$
$$\geqslant d_{(2n+1)^d-1}(E_{v,m};L_p(\mathbb{R}^d))$$
$$\geqslant d_{(2n+1)^d-1}(M_{(2n+1)^d};L_p(\mathbb{R}^d))$$
$$\geqslant C_2(\hbar)^{-m}N^{-m/d}.$$

Thus we get the lower estimates of Theorem 1 and the proof of Theorem 1 is complete.

From Lemma 2, using the same methods as in the proof of Theorem 1, we can prove Theorem 2, and the details are omitted.

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